VARIATIONS AND GENERALIZATIONS OF THE DEHN FUNCTION

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0. Notations

Here are some notations we will use throughout this survey.

(1) For group elements $g, h \in G$, $g^h = h^{-1}gh$. And $[g, h] = g^{-1}h^{-1}gh$.
(2) Let X be a finite set, then $F(X)$ is the free group generated by X. It is also represents all reduced words in alphabet $X \cup X^{-1}$. The length function $|\cdot|_X : F(X) \to \mathbb{N}$ maps a reduced word to its length. The set of words of the alphabet X (no inverses included) is denoted by X^*.

1. The Dehn function

Let G be a finitely presented group. Consider a finite presentation of G,

$$\mathcal{P} = \langle x_1, \dots, x_k \mid r_1, \dots, r_m \rangle.$$

The *area* of a word w that represents the identity element in G, is the minimal integer l such that there exist group elements $f_1, f_2, \dots, f_l$ in G such that

$$w =_{F(X)} \prod_{j=1}^{l} r_{i_j}^{\varepsilon_j f_j}, \varepsilon_j \in \{\pm 1\}, i_j \in \{1, 2, \dots, m\}. \tag{1}$$

We denote by $\text{Area}_{\mathcal{P}}(w)$ the area of w with respect to the presentation $\mathcal{P}$.

Definition 1.1. We define the *Dehn function* with respect to the finite presentation $\mathcal{P}$, denoted by $\delta_{\mathcal{P}}(n)$, as

$$\delta_{\mathcal{P}}(n) := \max\{\text{Area}_{\mathcal{P}}(w) \mid |w|_X \leqslant n\}.$$

To understand the asymptotic behaviour of the Dehn function, we introduce the following partial order and equivalence relation on functions of natural numbers.

Definition 1.2. Let $f, g : \mathbb{N} \to \mathbb{N}$ be non-decreasing functions. We say f is *asymptotically bounded above* by g, denoted by $f \preccurlyeq g$, if there exists a constant C such that

$$f(n) \leqslant Cg(Cn) + Cn + C, \forall n \in \mathbb{N}.$$

In addition, we say f is *asymptotically equivalent* to g , denoted by $f \cong g$, if $f \preccurlyeq g$ and $g \preccurlyeq f$.

Gromov [Gro87] showed that *the Dehn function* of a finitely presented group does not depend on the choice of the finite presentation up to equivalence. In fact, the Dehn function is a quasi-isometry invariant. Thus we define the Dehn function of a finitely presented group G to be the Dehn function of any its finitely presentations, denoted by δ_G.

A well-known result shows that the Dehn function of a finitely presented group characterises the decidability of the word problem.

Theorem 1.3 (Madlener-Otto [MO85]). *Let G be a finitely presented group. Then the word problem of G is decidable if and only if $\delta_G(n)$ is sub-recursive.*

We refer [Bri02] for a detail survey of the Dehn function.

2. The centralized and abelian Dehn function

Again, let G be a finite presented group with presentation $\mathcal{P} = \langle X \mid R \rangle$. Then G is isomorphic to $F(X)/N$, where N is the normal closure of R in $F(X)$. It is often much harder to estimate the lower bound of the Dehn function. So instead of considering representations of w that freely equal to w, we consider representations of (1) that equal to w modulo some normal subgroups in the free group. Two candidates are $[F, N]$ and $[N, N]$. When we modulo $[F, N]$, we only count the net number of use of every relators, while in the case of $[N, N]$, we are able to gather relators conjugated by the same element together.

Formally, the centralized area of a word w, named after the reason that we let all relators be in the center, is the minimal integer l such that there exist $i_1, i_2, \ldots, i_m \in \mathbb{Z}$ such that

$$w \equiv \prod_{j=1}^{m} r_j^{i_j} \mod [F, N], \sum_{j=1}^{m} |i_j| = l.$$

Thus the centralized Dehn function with respect to the presentation $\mathcal{P}$ is defined to be

$$\delta_{\mathcal{P}}^{\text{cent}}(n) := \max\{\text{Area}_{\mathcal{P}}^{\text{cent}}(w) \mid |w|_X \leqslant n\}.$$

The centralized Dehn function is independent from the finite presentation we choose [BMS93]. Thus the centralized Dehn function $\delta_G^{\text{cent}}(n)$ of a group G is well-defined.

We will still discuss the centralized Dehn function under the equivalence $\cong$, as we will say the function is linear, quadratic etc. But the centralized Dehn function is not a quasi-isometry invariant. In particular, Baumslag, Miller, Short gave an examples of quasi-isometric groups G and H (one has index 2 in the other) such that $\delta_G^{\text{cent}} \ncong \delta_H^{\text{cent}}$.

The centralized Dehn function gives the lower bound for the Dehn function by the definition, and thus it becomes a very useful tool for estimating the Dehn function from below.

The following results show that centralized Dehn function and Dehn function sometimes agree.

Theorem 2.1 (Baumslag-Miller-Short [BMS93]). *For every integer $l > 0$, there is a finitely presented group G_l such that $\delta_{G_l}^{\text{cent}}(n) \cong \delta_{G_l}(n) \cong n^l$.*

Using similar method, we have

Theorem 2.2 (Baumslag-Miller-Short [BMS93]). *There is a metabelian polycyclic group G of Hirsch length 3 such that $\delta_G^{\text{cent}}(n) \cong \delta_G(n) \cong 2^n$.*

In connection with the centralized isoperimetric function δ_G^{cent} it is useful to recall the following exact sequence:

$$\begin{array}{ccccccccc}
0 \longrightarrow & N \cap [F,F]/[N,F] & \longrightarrow & N/[N,F] & \longrightarrow & F/[F,F] & \longrightarrow & F/N[F,F] & \longrightarrow 0 \\
& \| & & \| & & \| & & \| & \\
0 \longrightarrow & H_2G & \longrightarrow & (H_1R)_G = N/[N,F] & \longrightarrow & H_1F & \longrightarrow & H_1G & \longrightarrow 0
\end{array}$$

This sequence is the usual five term exact sequence in the homology of groups associated with the presentation $1 \to N \to F \to G \to 1$. Since $F/[F,F]$ is a finitely generated free abelian group, it follows that $N/[N,F]$ splits as a direct sum $N/[N,F] \cong H_G \oplus \mathbb{Z}^k$ for some $k \geqslant 0$. The H_2G is the more interesting part since $\mathbb{Z}^k$ makes only a linear contribution.

For finitely generated nilpotent groups, the centralized Dehn function is particularly useful. R. Young showed that it can be realised by a distortion function of a central extension of G by $\mathbb{Z}$.

Theorem 2.3 (Young [You08]). *If G is a finitely generated nilpotent group, A is a finitely generated abelian group,*

$$0 \to A \to H \to G \to 1$$

is a central extension of G, and k is the degree of distortion of A in H, then if $k \geqslant 2$ we have $\delta_G^{\mathrm{cent}}(n) \succcurlyeq n^k$.

Conversely, if G is a finitely generated nilpotent group and $\delta_G^{\mathrm{cent}}(n) \cong n^k$, then there exists a central extension

$$0 \to \mathbb{Z} \to H \to G \to 1$$

such that $\Delta_{\mathbb{Z}}^H \cong n^k$.

And the distortion of an abelian subgroup in a nilpotent group can be computed by a theorem of D. Osin [Osi01].

We now consider equation (1) modulo $[N,N]$. Let G be a finitely presented group with a finite presentation $\mathcal{P} = \langle X \mid R \rangle$. We denote by N the normal closure of R in $F(X)$. For a word w that represents the identity, the minimal l such that

$$w \equiv \prod_{j=1}^{l} r_{i_j}^{\varepsilon_j f_j} \mod [N,N], \varepsilon_j \in \{\pm 1\}, i_j \in \{1,2,\ldots,m\}.$$

holds is the abelian area of w, denoted by $\mathrm{Area}_{\mathcal{P}}^{\mathrm{ab}}(w)$. Then the abelian Dehn function of G with respect to the presentation $\mathcal{P}$ is

$$\delta_{\mathcal{P}}^{\mathrm{ab}}(n) = \max\{\mathrm{Area}_{\mathcal{P}}^{\mathrm{ab}}(w) \mid w =_G 1, |w|_X \leqslant n\}.$$

Unlike the centralized Dehn function, the abelian one is a quasi-isometry invariant.

Theorem 2.4 (Baumslag-Miller-Short [BMS93]). *Let G be a finitely presented group. If H is a group that is quasi-isometric to G, then H is finitely presented and $\delta_G^{\mathrm{ab}}(n) \cong \delta_H^{\mathrm{ab}}(n)$.*

By definition, we immediately have $\delta_G^{\mathrm{cent}}(n) \preccurlyeq \delta_G^{\mathrm{ab}}(n) \preccurlyeq \delta_G(n)$. Note that the abelian Dehn function has a close connection to the relation module $N^{\mathrm{ab}} = [N,N]$. If the structure of N^{ab} is well-understood, then the abelian Dehn function can be computed. For example, the finitely generated torsion-free one-relator groups is known to have free relation module of rank one. In fact, the relation module is free if and only if the group G have aspherical presentation (i.e., the presentation 2-complex is aspherical). Applying this idea we have

Theorem 2.5 (Baumslag-Miller-Short [BMS93]). *Suppose G is the amalgamated free product of two finitely generated free group with a cyclic amalgamated subgroup generated by proper powers in each fact, i.e., G has the following presentation*

$$G = \langle a_1,\ldots,a_s,b_1,\ldots,b_t \mid u^p = v^q \rangle, u \in F(a_1,\ldots,a_s), v \in F(b_1,\ldots,b_t), p,q \geqslant 2.$$

Then $\delta_G(n) \cong \delta_G^{\mathrm{ab}}(n) \cong n^2$.

This close relationship of the abelian Dehn function to the homology of G will enable us to extent it to a much wider class of groups later.

3. The averaged Dehn function

The Dehn function measures the extreme case of the complexity to decompose a word represents the identity to a product of conjugates of relators. Gromov proposed a variation of this concept: what the average complexity is, taken over all words represent the identity. He claimed that this averaged Dehn function should be strictly asymptotically smaller than the Dehn function.

Let $G = \langle x_1, \dots, x_k \rangle$. Suppose we have a probability distribution function on the set of words (not necessarily reduced) that represent the identity. Then we can define the averaged Dehn function as following

$$\delta_G^{\text{avg}}(n) = E_{\overline{p}_n}(\text{Area}(w)) = \sum_w \text{Area}(w) p(w).$$

There are many ways to put a probability measure on the set of words that represent the identity. The random walk is the most obvious choice (provided by the fact there are a lot of work on random walks of groups). One way to do this is to consider a symmetric random walk on the generator (including the identity e). That is, if our group G has a generating set $x_1, x_2, \dots, x_k$, the probability measure p is defined to be

$$p(g) = \begin{cases} \frac{1}{2k+1} & g = e \text{ or } g = x_i^{\pm 1}, \\ 0 & \text{otherwise.} \end{cases}$$

Though any probability measure with $p(e) > 0$ and $p(x) = p(x^{-1})$ will serve the purpose. We use p to construct random walk on the group where $p^{(n)}(x)$, the nth convolution power of p, is the probability that an n-step random walk starting at e ends at x. We also define $p^{(n)}(x, y) \equiv p^{(n)}(x^{-1}y)$, the probability of going from x to y in n steps and $p(x, y) \equiv p^{(1)}(x, y)$.

We normalize $p_n \mid_{g_1 g_2 \cdots g_n = e}$, which is nonzero, to a probability measure $\overline{p}_n$. Its support is the set of all lazy words of length n which are the identity in G. The averaged Dehn function is defined to be

$$\delta_G^{\text{avg}}(n) = E_{\overline{p}_n}(\text{Area}(w)) = \sum_w \text{Area}(w) \overline{p}_n(w).$$

Another way to consider a symmetric random walk on the set $X \cup X^{-1}$, or equivalently put a probability measure is to put an even distribution to all words $w \in (X \cup X^{-1})^*$ (possible non-reduced) that $w =_G 1$. For every positive integer n, we define the sets

$$B_G(n) = \{ w \in (X \cup X^{-1})^* \mid w =_G 1, |w|_X \leqslant n \}.$$

In this case, we define the averaged Dehn function as

$$\delta_G^{\text{avg}}(n) = \frac{\sum_{w \in B_G(n)} \text{Area}\, w}{|B_G(n)|}.$$

Note that the $\overline{p}_n$ we defined above is in fact the even distribution on the set

$$B'_G(n) = \{ w \in (\{e\} \cup X \cup X^{-1})^* \mid w =_G 1, |w|_X \leqslant n \}.$$

It is unknown if the averaged Dehn function is invariant under quasi-isometry or even change of generators.

Those following theorems support Gromov's claim:

Theorem 3.1 (Bogopolski-Ventura [BV08]). *Let $X = \{x_1, x_2, \dots, x_k\}$, and $G = \langle X \mid R \rangle$ be a finite presentation of an abelian quotient of $F(X)$. Then,*

$$\delta_G^{\mathrm{avg}}(n) = O(n(\ln n)^2).$$

Theorem 3.2 (Young [You08]). *If G is a finitely generated nilpotent group with Dehn function $\delta_G(n) = O(n^k)$, then if $k > 2$, its averaged Dehn function for any presentation satisfies $\delta_G^{\mathrm{avg}}(n) = O(n^{k/2})$, and if $k = 2$, $\delta_G^{\mathrm{avg}}(n) = O(n \ln n)$.*

Note that the above theorem does not include the case that $k = 1$, where G is virtually $\mathbb{Z}$.

One might argue that using the random walk method, we have to count a short word that represents the identity many times, which makes the sub-asymptotical behaviour less impressive. We can consider a uniform distribution on the set

$$W_n = \{w \mid w \in F(X), w =_G 1, |w|_X \leqslant n\}.$$

Thus we can define the averaged Dehn function as

$$\delta_G^{\mathrm{avg}}(n) = \frac{\sum_{w \in W_n} \mathrm{Area}(w)}{|W_n|}.$$

But in general, the set W_n is less understood. For cases like one relator groups and the free abelian group of rank 2, it is possible to estimate the averaged Dehn function for this probability model.

M. Sapir induced another notion called the random Dehn function. Let G be a finitely presented group and and $T(G)$ a geodesic combing (i.e. for every element $g \in G$, we associate it with a unique geodesic γ_g from e to g). We define the area of a word $w \in (X \cup X^{-1})^*$, not necessarily represents the identity, to be

$$\mathrm{Area}(w) = \mathrm{Area}(w\gamma_w^{-1}).$$

Note that this area agrees with our previous definition on the set of words represent the identity.

A function $f : \mathbb{N} \to \mathbb{N}$ is random isoperimetric function for G if

$$\frac{|\{w \in (X \cup X^{-1})^* \mid |w|_{X^*} \leqslant n, \mathrm{Area}(w) \geqslant f(n)\}|}{|w \in (X \cup X^{-1})^* \mid |w|_{X^*} \leqslant n\}|}$$

The random Dehn function for G is the smallest random isoperimetric function, denoted by $\delta_G^{\mathrm{rd}}(n)$. It depends on the choices of the presentation and geodesic combing. M. Sapir claimed that for any finite presentation of an abelian group G and for any combing $\delta_G^{\mathrm{ab}}(n) \preccurlyeq n \log n$.

4. The homological and homotopical Dehn function

The standard Dehn function reflects isoperimetric property of the spaces (we consider simply connected Riemannian manifolds or simplicial complexes) on which the group acts geometrically [BP94]. One immediate way to generalise it is to extend this idea to higher dimensions. But the standard definition of Dehn function does not involving higher dimensional structure. Thus we have to give an equivalent definition of the Dehn function that has potential to generalise to a higher definition.

First let us recall the definition of finiteness properties of groups. For a group G, a classifying space $K(G, 1)$ is a connected space such that $\pi(K(G, 1)) = G$ and its universal cover is contractible.

Definition 4.1. A group G is said to be

(1) of type F_n for a natural number n if its classifying space $K(G,1)$ has finite n-skeleton, or equivalently G acts geometrically on some $(k-1)$-connected CW-complex;
(2) of type F_∞ if it is of type F_n for every $n \in \mathbb{N}$;
(3) of type F if $K(G,1)$ is finite.

It is time to give a new and more geometric definition of the Dehn function. In essence, we now fill a manifold just as we fill a loop in Cayley 2-complex and we will properly define what is the meaning of "area (or volume in higher dimension)" for those fillings. But we will only focus on fill nicer manifolds. This leads to us to define the admissible maps as following.

Definition 4.2. Let W be a compact k-manifold and X a CW-complex. An admissible map from W to X is a map $f : W \to X^{(k)}$ such that $f^{-1}(X^{(k)} \setminus X^{(k-1)})$ is a disjoint union of open k-dimensional balls in W, each mapped homeomorphically to a k-cell of X. We define the $\operatorname{vol} f$ of f as the number of these balls.

Admissible maps are abundant, in the sense that every continuous map is homotopic to an admissible map [BBFS09].

Let X be a k-connected (i.e. $\pi_i(X) = 0$ for $1 \leqslant i \leqslant k$) CW-complex and $\alpha : S^k \to X$ be an admissible map, then the filling volume of α is

$$\mathrm{FVol}_X^{(k)}(\alpha) = \inf_\beta \{\operatorname{vol}\beta \mid \beta : D^{k+1} \to X, \beta\,|_{S^k} = \alpha, \beta \text{ is admissible}\}.$$

Then the k-dimensional Dehn function of X is defined to be

$$\delta_X^{(k)}(n) = \sup_\alpha \mathrm{FVol}_X^{(k)}(\alpha),$$

where α ranges all admissible maps satisfying $\operatorname{vol}\alpha \leqslant n$.

Let G be a finitely presented group of type F_{k+1} and X is a $K(G,1)$ space. Then G acts on the universal cover $\tilde{X}$ geometrically (that is, coboundedly and properly-discontinuously). We define the k-dimensional homotopical Dehn function of group G to be the Dehn function of the space X. As Gromov pointed out and Bridson proved, the homotopical Dehn function is equivalent to the Dehn function [BP94] when $k = 1$. Thus, we usually omit the superscript and denote $\delta_G(n) = \delta_G^{(1)}(n)$.The following proposition shows that this notion of homotopical Dehn function is well-defined and it is a quasi-isometry invariant.

Proposition 4.3 (Alonso-Wang-Pride [AWP99])**.** Let G be a group of type F_{k+1}, then $\delta_G^{(i)}(n)$ for $1 \leqslant i \leqslant k$ is well-defined. Moreover, if H is quasi-isometric to G then H is also of type F_{k+1} and $\delta_G^{(i)}(n) \cong \delta_H^{(i)}(n)$ for $1 \leqslant i \leqslant k$.

When $k = 2$, the second order Dehn function was studied extensively by many mathematicians. Bogley and Burton shows that each hyperbolic group has linear second order Dehn function [ABB$^+$98], though the converse is false since any group with a finite aspherical presentation has linear second order Dehn function (e.g. $\mathbb{Z}^2$). The synchronously automatic groups and semihyperbolic groups have quadratically bounded Dehn functions while the asynchronously automatic groups have exponentially bound Dehn functions. Moreover, if G is synchronously combable then $\delta_G^{(2)}(n) \leqslant 2^n$ [Wan96]. Note that synchronously combable groups, semihyperbolic groups have quadratically bounded ordinary Dehn function and asynchronously automatic groups as well as asynchronously combable groups have exponentially bounded Dehn function [ECH$^+$92]. There are also results investigating the behaviour of the second order Dehn functions under taking direct products, amalgamated free products and HNN-extensions [WP99], [WB09]. Those results show the second order Dehn function shares a lot of similarities with the Dehn function.

The real differences start to reveal as Wang showed that $\delta_{\mathbb{Z}^k}(n) \cong n^{3/2}$ when $k \geqslant 3$ [Wan02]. This falls into the famous isoperimetric gap of the Dehn function, suggesting the isoperimetric spectrum should be different for second order Dehn function and presumably for higher dimensional homotopical Dehn function, which is indeed the case. The kth isoperimetric spectrum is defined to be

$$\mathrm{IP}^{(k)} = \{\alpha \mid \text{ there exists } G \text{ of type } F_{k+1} \text{ such that } \delta_G^{(k)} \cong n^\alpha\}.$$

The higher Dimensional Dehn functions of snowflakes groups give the following result.

Theorem 4.4 (Brady-Bridson-Forester-Shankar[BBFS09])**.** *The closure of* $\mathrm{IP}^{(k)}$ *contains* $\{1\} \cup [\frac{k}{k+1}, \infty)$.

The full picture of isoperimetric spectrum is still unknown even for $k = 2$. A. Mukherjee showed that the second order Dehn function of $\mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}$ is bounded by $n \ln n$ when no eigenvalues of φ is ± 1 [Muk16] though no lower bound is obtained. Also it is unclear if groups constructed by Birget, Rips, Sapir and later by Ol'shanskiy will give a finer detail of $\mathrm{IP}^{(k)}$ [BOsRS02].

Another important difference is that $\delta_G^{(2)}(x)$ is always sub-recursive [Pap00], no matter the decidability of the word problem (D. Collins and C. Miller gave an example of group of type F with undecidable word problem and aspherical presentation [CM99]), while R. Young constructed a group with non-subrecursive k-dimensional Dehn function [You11] for $k \geqslant 3$. It will be interesting to understand the situation for higher dimensional Dehn functions.

Next we define the homological version of the Dehn function. One might notice that the abelian Dehn function already has its connection to homology. We will discuss the connection later. First let us recall the definition of algebraic finiteness properties.

Definition 4.5. A group G is said to be

(1) of type FP_n if there exists an exact sequence of the form

$$P_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0,$$

where P_i are finitely generated projective $\mathbb{Z}G$-modules, or equivalently G acts geometrically and cellularly on some homologically $(n-1)$-connected space;

(2) of type FP_∞ if it is of type FP_n for every $n \in \mathbb{N}$;

(3) of type FP if there exists an exact sequence of the form

$$0 \to P_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0,$$

where P_i are finitely generated projective $\mathbb{Z}G$-modules.

Note that the set of groups of type FP_2 is strictly larger than the set of finitely presented groups (equivalently, groups of type F_2) [BB97]. And in fact, the set of groups of type FP_2 is uncountable [Lea18].

For homological version of the Dehn function, we would like to fill k-cycles by $(k+1)$-chains. This naturally requires us to find a space on which G acts geometrically and cellularly and has trivial kth homology group. We start with the case when $k = 1$.

Let X be a CW-complex such that $H_1(X) = 0$. Let $\gamma \in Z_1(X)$ be a 1-cycle, then there exists a 2-chain $c \in C_2(X)$ of the form $c = \sum_i a_i\sigma_i$, where $a_i \in \mathbb{Z}$ and σ_i are 2-cells, such that $\gamma = \partial c$. The homological filling area of γ is

$$\mathrm{HArea}_X(\gamma) = \min\{\|c\|_1 \mid \partial c = \gamma\},$$

where $\|\cdot\|_1$ is the l_1-norm. Then the homological filling function of X is

$$\mathrm{FA}_X(n) = \sup\{\mathrm{HArea}_X(\gamma) \mid \gamma \in Z_1(X), \|\gamma\|_1 \leqslant n\}.$$

One can also extend the definition of the abelian Dehn function as follows:

$$\delta_X^{\mathrm{ab}}(n) = \sup\{\mathrm{HArea}_X(\gamma) \mid \gamma \text{ is a loop}, \|\gamma\|_1 \leqslant n\}.$$

By this definition, we immediately have $\delta_X^{\mathrm{ab}}(n) \preccurlyeq \mathrm{FA}_X(n)$.

As previous discussion, for a group G we can define its second order homological Dehn function as the homological filling function of a space X that G acts geometrically and cellularly. The existence of such space is provided by the following lemma:

Lemma 4.6 (Brady-Kropholler-Soroko [BKS21]). *Let G be a group of type FP_2, then there exists a CW-complex X with $H_1(X) = 0$ such that G acts on it freely, cellularly, cocompactly and has one orbit of vertices.*

Therefore we define the second order homological Dehn function and abelian Dehn function respectively as

$$\mathrm{FA}_G(n) := \mathrm{FA}_X(n), \text{ and } \delta_G^{\mathrm{ab}}(n) := \delta_X^{\mathrm{ab}}(n).$$

Though the construction of X depends on the presentation, as Brady, Kropholler and Soroko showed, both Dehn functions are independent of the choice of presentations and are quasi-isometry invariant [BKS21].

Recall that a function $f : \mathbb{N} \to [o, \infty)$ is *super-additive* if $f(m + n) \geqslant f(m) + f(n)$ for $m, n \in \mathbb{N}$. For a function $f : \mathbb{N} \to [o, \infty)$, its super-additive closure, denoted by $\bar{f}$, is defined to be the smallest super-additive function that bounds above f. Or equivalently,

$$\bar{f}(n) = \max\{f(n_1) + \cdots + f(n_r) \mid r \geqslant 1, n_i \in \mathbb{N}, \sum_{i=1}^{r} n_i = n\}.$$

Super-additivity was first recognized as a useful property in connection with Dehn functions by S. Brick [Bri93] who called it "subnegativity", and it has since then been studied by V. Guba and M. Sapir [GS99]. For example, Dehn functions of free products behaves nicely if they are supper-additive, and the super-additivity also relates to the whole picture of the isoperimetric spectrum of the Dehn function [BOsRS02]. It is conjectured by Guba and Sapir that the Dehn function of a finitely presented group is equivalent to a super-additive function otherwise G and $G * \mathbb{Z}$ will have different Dehn function up to $\cong$. The homological version of this conjecture has already been proved.

Theorem 4.7 (Brady-Kropholler-Soroko [BKS21]). *Every second order homological Dehn function is equivalent to a super-additive function.*

As for finitely presented groups, their second order homological Dehn function is in fact bounded above the super-additive closure of their abelian Dehn function, that is, for a finitely presented group G, $\mathrm{FA}_G(n) \preccurlyeq \bar{\delta}_G^{\mathrm{ab}}(n)$ [BKS21]. It was also shown earlier by Gersten that $\mathrm{FA}_G(n) \leqslant \bar{\delta}_G(n)$ [Ger92].

The relation between the second order Dehn function and Dehn function is still unknown. It is conjectured that the second order homological Dehn function is bounded above by the Dehn function. The following result shows that they can be different.

Theorem 4.8 (Abrams-Brady-Dani-Young [ABDY13])**.** *Let $f(n)$ be 2^n or n^d for sufficiently large d, then there is a finitely presented group G such that*

$$\mathrm{FA}_G(n) \preccurlyeq n^5, \delta_G(n) \succcurlyeq \delta_G(n).$$

It also follows that the abelian Dehn function and Dehn function can be different.

Still, the second order Dehn function resembles many properties of the Dehn function. First, it is a quasi-isometry invariant [BKS21]. Secondly, it also characterises the hyperbolic groups as follows

Theorem 4.9 (Gersten [Ger92])**.** *A finitely presented group is hyperbolic if and only if its second order homological Dehn function is linear.*

The trade-off, as one might expect, of generalising the Dehn function to a larger class of groups is that we no longer have connections between the homological Dehn function and complexity of the word problem. It is well-known that for a finitely presented group G a sub-recursive Dehn function $\delta_G(n)$ implies the solvability of the word problem of G and vice versa [MO85]. This nice property fails for homological Dehn function, as the following theorem shows.

Theorem 4.10 (Brady-Kropholler-Soroko [BKS21])**.** *There exists a group G of type FP_2 with $\mathrm{FA}_G(n) \cong n^4$ which has unsolvable word problem.*

But for a smaller class of groups, the classical decidability result is true to some extent. Let $\mathcal{F}_0$ be the set of the trivial group. The set $\mathcal{F}_{n+1}$ is inductively defined as the set of fundamental groups of a finite graph of $\mathcal{F}_n$ vertex groups and finitely generated free edge groups. And the homological Dehn function with coefficient $\mathbb{R}$ is defined as one expected. We have $\mathrm{FA}_G(n) \preccurlyeq \mathrm{FA}_{G,\mathbb{R}}(n)$ [Ger92].

Theorem 4.11 (Gersten [Ger92])**.** *If $G \in \mathcal{F}_n$, then G has a decidable word problem if and only if $\mathrm{FA}_{G,\mathbb{R}}(n)$ is sub-recursive.*

For $k \geqslant 2$, the definition of kth order homological Dehn function is straightforward. Let X be a homologically k-connected CW-complex and α a k-cycle. The homological filling volume of α is

$$\mathrm{HVol}_X^{(k+1)}(\alpha) := \inf\{\|\beta\|_1 \mid \beta \in C_{k+1}(X), \partial\beta = \alpha\}.$$

Thus the kth filling volume function of X is

$$\mathrm{FV}_X^{(k+1)}(n) = \sup_{\|\alpha\|_1 \leqslant n} \mathrm{HVol}_X(\alpha).$$

Let G be a group of type FP_{k+1} that acts geometrically on a homologically k-connected complex X. We define the kth order homological Dehn function for group G to be $\mathrm{FV}_G^{(k+1)}(n) := \mathrm{FV}_X^{(k+1)}(n)$. If X is also k-connected (or G is of type F_{k+1}), by Hurewicz theorem, one can replace cycles and chains by spheres and balls when $k \geqslant 2$. Thus it is not hard to show that $\delta_X^{(2)} \preccurlyeq \mathrm{FV}_X^{(k+1)}$ for $k = 2$ and $\delta_X^{(k)}(n) \cong \mathrm{FV}_X^{(k+1)}$ for $k \geqslant 3$. Consequently, if $k \geqslant 3$ and G is a group which acts geometrically on a k-connected complex, previous discussions imply that $\delta_G^{(k)} \cong \mathrm{FV}_G^{(k+1)}(n)$. R. Young showed that the bound can be strict for $k = 2$, that is,

Theorem 4.12 (Young [You11])**.** *There exists a finitely presented group G such that $\delta_G^{(2)} \ncong \mathrm{FV}_G^{(3)}(n)$*

The group can be take as direct product of two copies of $BS(1,2)$. Then $\delta_G^{(2)}(n) \preccurlyeq n^2$ and $\mathrm{FV}_G^{(3)} \succcurlyeq 2^{\sqrt{n}}$.

In addition to this, Young also shows that

Theorem 4.13 (Young [You11])**.** *There exist groups G_k for $k \geqslant 2$ such that $\mathrm{FV}_{G_k}^{(k)}$ is not sub-recursive.*

Thus there is no general relation between the decidability of the word problem and higher order homological Dehn function.

Another interesting topic is the homological isoperimetric spectrum, which can be defined as following.

$$\mathrm{HIP}^{(k+1)} = \{n^\alpha \mid \text{ there exists a group } G \text{ such that } \mathrm{FV}_G^{(k+1)} \cong n^\alpha\}.$$

Following the result of Theorem 4.4, the normal closure $\overline{\mathrm{HIP}}^{(k+1)}$ contains $\{1\} \cup [\frac{k}{k+1}, \infty)$ for $k \geqslant 3$. For the case $k = 1$, it has been shown by N. Brady, R. Kropholler and I. Soroko that the $\mathrm{HIP}^{(2)}$ is dense in $\{1\} \cup [4, \infty)$. It is very interesting to understand more of the set $\mathrm{HIP}^{(2)}$. The followings are two natural questions to ask

Question 4.14. Is the set $(1,2) \cap \mathrm{HIP}^{(2)}$ empty? In general, is the set $(1, \frac{k}{k+1}) \cap \mathrm{HIP}^{(k+1)}$ empty?

For $\mathrm{HIP}^{(2)}$, one could hope that the following might be true.

Question 4.15. Do we have an equality $\mathrm{HIP}^{(2)} = \{1\} \cup [2, \infty)$?

5. The Dehn function for infinite presentations

Let a finitely generated group G be defined by a presentation in terms of generators and defining relators

$$G = \langle X \mid R \rangle,$$

where $X = \{x_1, x_2, \ldots, x_k\}$ and R is a set of defining relators which are nonempty cyclically reduced words over the alphabet $X \cup X^{-1}$. The subset R is termed *decidable* (or recursive) if there is an algorithm to decide whether a given word over $X \cup X^{-1}$ belongs to R.

We then define the presentation complex for the group G with respect to the presentation $\langle X, R \rangle$. Let $K(X,R)$ be a 2-complex associated with the presentation so that $K(X,R)$ has a single 0-cell, oriented 1-cells of $K(X,R)$ are in bijextive correspondence with letters of $X \cup X^{-1}$, and 2-cells of $K(X,R)$ are in bijective correspondence with the words of R that naturally determine the attaching maps of the 2-cells. It is easy to see that the fundamental group $\pi_1(K(X,R))$ is isomorphic to G.

By a *van Kampen diagram* over the presentation $\langle X, R \rangle$ we mean a planar, finite, connected and simplt connected 2-complex Δ which is equipped with a continuous cellular map $\mu : \Delta \to K(X,R)$ whose restriction on every cell of Δ is a homeomorphism. By an *edge* of Δ we mean the closure of a 1-cell. If e is an oriented edge of Δ corresponds to a letter $x \in X \cup X^{-1}$, then a is termed the *label* of e and is denoted $\varphi(e)$. Note $\varphi(e^{-1}) = \varphi(e)^{-1}$, there e^{-1} denotes the edge with opposite orientation. If $p = e_1 \ldots e_m$ is a path in Δ, where $e_1, \ldots, e_m$ are oriented edges of Δ, then we set $\varphi(p) = \varphi(e_1) \ldots \varphi(e_m)$. The well-known van Kampen lemma states that a word w belongs to $\langle\langle R\rangle\rangle$ if and only if there exists a van Kampen diagram Δ over $\langle X \mid R\rangle$ whose boundary path $\partial\Delta$ is labeled by the word w. Let $\Delta(j)$ be the set of j-cells in Δ, $j = \{0, 1, 2\}$.

Let $w \in \langle\langle R\rangle\rangle$ and $j \in \{0, 1, 2\}$. Define $L_j(w)$ to be the minimal number of j-cells in a van Kampen diagram Δ over $\langle X, R\rangle$ whose boundary $\partial\Delta$ is labeled by the cyclic word w, that is

$$L_j(w) = \min\{|\Delta(j)| \mid \varphi(\partial\Delta) =_{F(X)} w.\}$$

Then we deine

$$\delta_j(n) = \max\{L_j(w) \mid w \in \langle\langle R\rangle\rangle, |w|_X \leqslant n\}.$$

If $j = 2$ and R is finite, $\delta_2(n)$ is the ordinary Dehn function. For $j = 0, 1, 2$, δ_j is referred to as the Dehn j-function of the specific presentation. It is obvious from the definition that such functions depends on the choice of presentation. The Dehn 2-function of any finitely generated group could be equal to 1 if we choose R to be all words equal to the identity.

When $j = 1$, the Dehn 1-function is actually equivalent to the derivation work function introduced by J. Birget. It measures the work of a derivation converting a word to the identity. More details can be found in [Bir98] and [Vac19].

When R is finite, all Dehn j-functions are non-distinguishable under $\cong$.

Theorem 5.1 (Grigorchuk-Ivanov [GI09])**.** *If G is finitely presented, equipped with the finite presentation $\langle X \mid R\rangle$, then all Dehn j-functions are equivalent under $\cong$ for $j = 0, 1, 2$.*

Some immediate results follow the definition.

Theorem 5.2 (Grigorchuk-Ivanov [GI09])**.** *Let $\langle X \mid R\rangle$ be a presentation of G such that R is decidable. Then the word problem for G is solvable if and only if the Dehn 1-function $\delta_{1,G}(n)$ is computable.*

The statement is not true for the Dehn 2-function. According to [GI09, Example 2.4], there is a finite generated infinite presented group with the set of relators being decidable that has undecidable word problem but a recursive Dehn 2-function.

Theorem 5.3 (Grigorchuk-Ivanov [GI09])**.** *Let a group $G = \langle x_1, \ldots, x_k\rangle$ be generated by elements $x_1, \ldots, x_k$. Then the word problem for G is in* **NP** *if and only if there exists a presentation $\langle x_1, \ldots, x_k \mid R\rangle$ for G such that its Dehn 1-function δ_1 is bounded by a polynomial and the problem to decide whether a word w belongs to R is in* **NP**.

While there is no δ_2 for finite presentations between n and n^2 [Ol'92], this is not the case for infinite presentations.

Theorem 5.4 (Grigorchuk-Ivanov [GI09])**.** *Let $m \geqslant 2, l \geqslant 2^{48}$ and l be either odd or divisible by 2^9. Let $B := B(m, l)$ be the free Burnside group equipped with presentation given by Ol'shanskiy (refer to* [Ol'91]*). Then the Dehn 1-function $\delta_{1,B}(n)$ is bounded from above by $n^{19/12}$. In addition, $\delta_{0,B}(n) \leqslant 2x^{19/12}$ and $f_{2,B} \leqslant \frac{2}{l}x^{19/12}$.*

As a consequence of the above theorem, we have

Corollary 5.5. *Let $m \geqslant 2, l \geqslant 2^{48}$ and l be either odd or divisible by 2^9. Then the word and conjugacy problems for the free Burnside group $B(m, l)$ are in* **NP**.

In an ongoing work by Osin and Rybak, they present a different way to generalise the Dehn function to finitely generated infinite presented groups.

Let $G = \langle X \rangle$ be a finitely generated group. We define

$$S_k = \{w \in F(X) \mid |w|_X \leqslant k, w =_G 1\}$$

and

$$G_k = \langle X \mid S_k \rangle.$$

Then G is the direct limit of G_ks, i.e., $G = \lim_{k\to\infty} G_k$. In addition, let $\Pi = \{(k, m, n) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \mid m \geqslant k\}$.

The *isoperimetric spectrum* $f_{G,X} : \Pi \to \mathbb{N}$ of a finitely generated group G is

$$f_G(k, m, n) = \max\{\text{Area}_{S_m}(w) \mid w \in \langle\langle S_k \rangle\rangle, |w|_X \leqslant n\}.$$

We introduce an analogue of $\cong$ to this multivariable function. For functions $f, g : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$,

$$f \preceq g \text{ if there exists } C > 0 \text{ such that } f(k, Cm, n) \leqslant Cg(Ck, m, Cn) + C\frac{n}{m} + C, \forall k, m, n.$$

And $f \sim g$ if $f \preceq g$ and $g \preceq f$.

Theorem 5.6 (Osin-Rybak)**.** *Let G, H be finitely generated groups such that G is quasi-isomorphic to H. Then*

$$f_G(k, m, n) \sim f_H(k, m, n).$$

Note that if G is finitely presented, $G_k \cong G$ for all k greater than a certain number. Thus the asymptotic behaviour of the isoperimetric spectrum does not depend on k.

Theorem 5.7 (Osin-Rybak)**.** *Let G be a finitely presented group, then*

$$\frac{\delta_G(n)}{\delta_G(m)} \preceq f_{G,X}(k, m, n) \preceq \frac{\delta_G(n)}{m}.$$

It follows that

Corollary 5.8. *(1) If G is hyperbolic, then $f_{G,X}(k, m, n) \sim \frac{n}{m}$.*

(2) If G is semi-hyperbolic, then $f_{G,X}(k, m, n) \sim \frac{n^2}{m^2}$.

Definition 5.9. The isoperimetric spectrum is *strongly linear* if $f \sim \frac{n}{m}$ and is *weakly linear* if the constant C depends on k.

We have

Theorem 5.10 (Osin-Rybak)**.** *For the following groups, the isoperimetric spectrum is strongly linear.*

(a) $K \wr \mathbb{Z}, |K| < \infty$;
(b) finitely generated infinite presented $C'(1/6)$ groups;
(c) free burnside groups $B(m, n)$ for $n > 10^{10}$ odd.

The last generalization of Dehn function involving infinitely presented group is the *verbal Dehn function* introduced by Ol'shanskiy and Sapir. Let $\mathcal{V}$ be a variety that is defined by a finite set of identities, then it can be defined by a single law $v = 1$ for some word $v = v(x_1, x_2, \dots, x_k)$ from the (absolute) free group $F(X_1, X_2, \dots, X_n, \dots)$ of infinite rank. Let $V \leqslant F$ be the verbal subgroup consisting of all words vanishing in all groups of the variety $\mathcal{V}$. THen $w \in V$ can be written as the product

$$\prod_{i=1}^{N} u_i^{-1} v(X_{i1}, X_{i2}, \dots, X_{im})^{\pm 1} u_i.$$

A function $f_V : \mathbb{N} \to \mathbb{N}$ is the verbal isoperimetric function of the word v if for any word $w \in V$, there exists a representation as above such that $\sum_{ij} |X_{ij}| \leqslant f_V(|w|)$. The smallest f_V is called the *verbal Dehn function* of $\mathcal{V}$, denoted by $\delta_{\mathcal{V}}^{\text{verb}}$. It can be shown that $\delta_{\mathcal{V}}^{\text{verb}}$ does not depends on the choice of v under the equivalence $\cong$.

Proposition 5.11 ([OS00])**.** $\delta_{\mathcal{V}}^{\text{verb}}$ is superadditive.

The verbal Dehn function is useful when embedding relative free groups into finitely presented groups.

Theorem 5.12 (Olshankiy-Sapir [OS00])**.** *Let $f(n)$ be the verbal isoperimetric function of a group variety $\mathcal{V}$ defined by $v = 1$. Then the relative free group $F_m(\mathcal{V})$ of rank m in the variety $\mathcal{V}$ can be isomorphically embedded into a finitely presented group $H = H(v, m)$ with an isoperimetric function $n^2 f(n^2)^2$.*

One example is the Burnside variety $\mathcal{B}_n$ for n sufficiently large. In this case, $f(s) = s^4$ is a verbal isoperimetric function [Ol'91]. Thus free Burnside group for n sufficiently large can be embedded into a finitely presented group of Dehn function bounded above by n^18. In an unpublished work of R.Mikhalov, the verbal isoperimetric function can be chose to be $n^{1+\varepsilon(n)}$ where $\varepsilon(n) \to 0$ as $n \to \infty$. Thus the result can be improved to $n^{8+\varepsilon(n)}$. Though recently, F. Wagner improved the embedding such that Free Burnside groups (for n sufficiently large) can be embedded into groups with quadratic Dehn function [Wag20].

For solvable varieties, by the work of Kleiman, there exists a solvable variety with non-recursive verbal Dehn function [Kle82].

6. Conjugacy Length Function

For the conjugacy problem, we consider a different function, the conjugator length function, since we are more interested in the length of the conjugator rather than the area of the annular diagram.

Let G be a group with a generating set $\{x_1, x_2, \ldots, x_k\}$. If g and h are conjugate elements of G, the *conjugacy distance* from g to h is

$$\mathrm{cd}(g, h) = \min\{|f|_X \mid f \in G, h = g^f\}.$$

Then we define *the conjugacy length function* to be

$$\mathrm{CLF}(n) = \max\{\mathrm{cd}(g, h) \mid g, h \in G \text{ are conjugate and } |g|_X + |f|_X \leqslant n\}.$$

It is not hard to see that the conjugacy problem of a group is solvable if and only if its conjugacy length function is recursive. Note that the solvability of the conjugacy problem does not pass to finite index subgroups or to finite extensions [CM77]. Moreover, CLF is dependent to the choice of the generating set but if CLF has a polynomial growth then the degree of the polynomial is independent of S. So CLF is not a quasi-isometry invariant.

There are many works in the literature giving various upper bounds for the conjugacy length function in certain classes of groups. Here we list a few of them.

Theorem 6.1. *(1) Hyperbolic groups have a linear upper bound* [BH99]*.*

(2) CAT(0) *groups have an exponential upper bound for conjugacy length* [BH99]*.*
(3) Free solvable groups have a cubic upper bound for CLF [Sal15]*.*
(4) Mapping class groups have a linear upper bound [Tao13]*.*
(5) Right-angled Artin groups have a linear upper bound [CGW09]*.*

(6) *Thompson's group F has a quadratic CLF* [BM21].
(7) *The wreath product of a finite abelian group A and a finitely generated abelian group of torsion-free rank $k > 0$ has exponential CLF if $k = 1$ and has CLF between 2^n and 2^{n^k} if $k > 1$* [FP21].

In an upcoming work of Bridson, Riley and Sale, they give examples with CLF equivalent arbitrary degree of polynomials and examples with CLF equivalent to 2^n.

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